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CONSONANCE/DISSONANCE – A HISTORICAL PERSPECTIVE

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ABSTRACT

This paper looks at the various approaches as taken in order to understand the phenomenon of consonance/dissonance. While an early understanding of consonance/dissonance was based upon number ratios (Pythagoras), it became a question of resonance for Descartes. Subsequently, Euler attempted to formalize the ratio approach by classifying intervals into consonance/dissonance classes according to prime factors producing impressive results. Helmholtz then somewhat picked up on where Descartes had left considering a physical/acoustical model based on roughness. Inevitable, the issue became a psychological one during the late 19th century with Stumpf which ultimately resulted in a cognitive model as produced by Hofmann-Engl in 1990. This model has remained unchallenged during the last 2 decades and has been found of good validity in a number of contexts.

1. INTRODUCTION

The phenomenon of consonance and dissonance has preoccupied thinkers over more than 2500 years. All in all, so the author suggests, there are 6 approaches to the issue. These approaches are in historical order: The number ratio theory, Descartes's theory, the prime factor theory, the roughness theory, the fusion theory and finally the cognitive theory. Not quite surprisingly, these approaches themselves have gained complexity in time but interestingly, they all appear to be floating about and being selected by individuals randomly and according to personal and intellectual preferences.

This paper will disseminate all six approaches with the intention to illustrate their strengths and weaknesses. However, this paper will go further by putting to rest those approaches which are so fundamentally flout that they can be dismissed with great certainty.

2. NUMBER RATIO THEORY

The link between consonance and number ratios has generally been attributed to Pythagoras (around 500 BC). The anecdote according to T. Stanley (1655) reports that Pythagoras's apprehending "came to him from God, as a most happy thing". He thence went into a shop testing a number of hammer against their sound properties. Here, he found that neither the shape nor the strength with which the hammers were struck had an impact on the quality of the sounds but the weight only. Following this discovery, he went home attached for equal strings on a beam across the room, attached four hammers at the end of the strings and tested the sound qualities of those hammers against each other. During this experiment, he realized that the diapason (octave) required hammers of the weight 1:2. This is the most consonant sound required the simplest number ratio.

Others, according to J. Hawkins (1776), attribute the discovery of consonance correlating to simple number ratios to Diocles, how so it has been suggested conducted a similar experiment using vessels of various magnitude. Disregarding the question of authorship, it is the following diagram (after Hawkins) which has been passed on through the the following millennia.

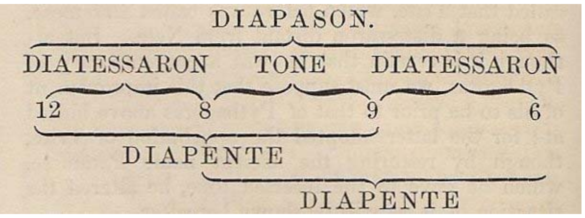

Figure 1: The number ratios and their correlating intervals attributed to Pythagoras.

Doubt about the correlation of ratios and consonance were raised by G. Galilei (1638) by contesting that other factors such as the tension and the thickness of a string too will have an impact on the intervals produced and a string of doubled thickness will not result in an interval of a diapason.

From a more contemporary view, there are two issues which are associated with the number ratio theory. Firstly, it is simply not clear what makes a simple ratio and what makes a complex ratio. This is, is 2:3 less or more complex then 3:4? As we will see, it was L. Euler (1739) who attempted to resolve this issue. However, a second issue might even be more pressing and this is the issue of categorical pitch interval perception as demonstrated by A. Houstma (1968). Here, we find that any ratio close to 1:2, such as 101:201 will be perceived just as a ratio as 1:2. In fact, that this is the case has been known to musicians and instrument tuners for centuries. This, however, has not prompted popular belief to hold onto the number ratio theory and even let J. Kepler to produce his famous *Harmonices Mundi* in 1619.

3. DESCARTES'S THEORY

Descartes writes in his Musicae Compendium:

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[quote]

Neque quis putet imaginarium illud, quod dicimus properie tantum ex divisione octave quintam generari & ditonum, ceteras per accidens, id enim eriam experientia compertum habeo in nervis testudinis vel alterius cujuslibet instrumenti, quorum unus si pulsetur vis ipsius soniconcutict omnes nervos qui aliquo genere quintae vel ditoni erunt acutiores, in iis autem qui quarta, vel alia consonantia distabant, id non siet: quae certe vis consonantiarum non nisi ex illarum perfectione potest oriri vel imperfectione, quae sic-licet primae per se consonantae sint, aliae autem per accidens, quia ex aliis neceessario fluunt.

[unquote]

[quote]

So that no one should think that this is imagination when we said that this strange division of the octave brings about the fifth and the major third and other random intervals, I found through experimentation that all other strings will resonate if a string a fifth or major third above has been plugged on lute or an other instruments. However, this does not happen if the interval is a fourth or any other consonance. Surely, this strength must be the result of the perfection or imperfection of these consonances, because the first consonances are per se consonances while the others consonances are random consonances and derived from the first consonances.

[unquote - translation: Hofmann-Engl, 2010]

Apparently, for the first time in history, Descartes refers to the resonance and this in order to underpin the concept of the special status of the fifth and the major third.

Contradicting himself somewhat, he subsequently arrives at the following figure:

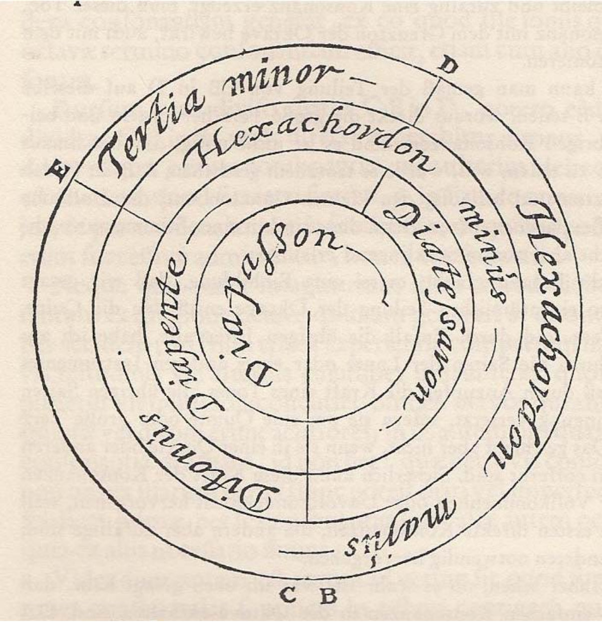


Figure 2: Descartes's classification of consonances as taken from his Musicae Compendium.

Here, the most perfect consonance is the octave, followed by the fifth and fourth at a second degree. At the third degree is the major third and minor sixth followed at fourth degree by the minor third and major sixth.

Whatever it exactly was that motivated Descartes to make this classification, it is supported by data as produced by Hofmann-Engl (2009).

4. PRIME FACTOR THEORY

The prime factor theory is a theory which was put forward by E. Euler in his *Tentamen novae theoriae musicae ex certissimis harmoniae principiis dilucide expositae.* Realizing that correlating the complexity of ratios to the degree of consonance cannot simply be decided by looking at a ratio, Euler felt compelled to map ratios onto a consonance degree value. This is:

$$f(n_1 : n_2) = d_c \qquad (1)$$

where *f* is a function, n_1 and n_2 are integers and d_c the consonance degree

Euler does not state this specifically but the text implies such a function. For the ratio 1 : *n*, P. Bailhache (1997) develops the following function in accordance with Euler's text:

$$f(1:n) = f(1 : p_1^{k_1} * p_2^{k_2} * \ldots * p_m^{k_m}) = \sum_{i=1}^{m} (p_i k_i - k_i) + 1 \qquad (2)$$

with *f* as a function, *n* as an integer and the ith prime factor component of n

In order to illustrate this function, we will give two examples. For the ratio of an octave we get:

$$f(1:2) = f(1:2^1) = (2*1-1)+1 = 2$$

and for the fictitious interval given as 1 : 240, we get:

$$f(1:240) = f(1:2^3*3^2*5^1) = [(2*3-3)+(3*2-2)+(5*1-1)]+1 = 12$$

Bailhache does not state explicitly the function for the case:

$$n_1 : n_2$$

However, the author describes the algorithm and applies it to an example. However, this general function is of the following form:

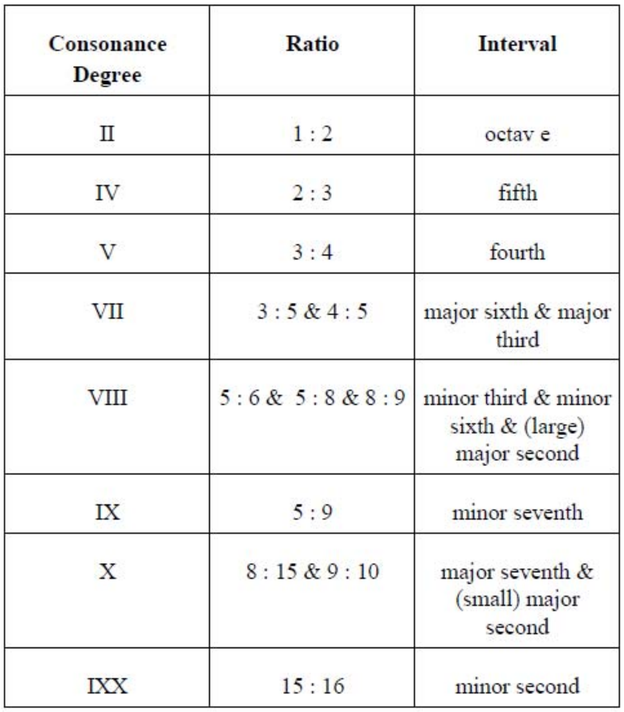
$$f(n_1:n_2) = f(HCF(n_{1,} n_2)) = f(LCM(HCF(n_{1,} n_2)))$$

$$= \sum_{1=i}^{m} (p_i^{k_i} - k_i) + 1 \; of \; LCM(HCF(n_{1,} n_2)) \qquad (3)$$

We will give a simple example for the fifth (2:3). The LCM(HCF (2, 3) = 6. Prime factors are 2 and 3, hence we get the consonance degree of:

$$f(2:3) = [(2*1-1)+(3*1-1)]+1 = 4$$

Now, applying this algorithm to the common intervals, we arrive at the following table:



Consonance Degree	Ratio	Interval
II	1 : 2	octave
IV	2 : 3	fifth
V	3 : 4	fourth
VII	3 : 5 & 4 : 5	major sixth & major third
VIII	5 : 6 & 5 : 8 & 8 : 9	minor third & minor sixth & (large) major second
IX	5 : 9	minor seventh
X	8 : 15 & 9 : 10	major seventh & (small) major second
IXX	15 : 16	minor second

Table 1: Classification of consonance degrees according to Euler

There is no question that this classification system is highly sophisticated, particularly when thinking about the time it was developed by Euler. Nevertheless, it is in conflict with Descartes's model and is not supported by the model as developed by Hofmann-Engl (1990).

5. ROUGHNESS THEORY

A different approach to the consonance/dissonance phenomenon was taken by H. Helmholtz (1877). Observing that the partials of two complex harmonic pitches at the distance of an octave will not produce any beats, he deducted that for other, more dissonant intervals, this will not be the case. This lead him to formulate the hypothesis that the more beats are contained within a sound, the more rough the sound will be and this roughness would result in the perception of increased dissonance. As far as the author is aware, Helmholtz actually never undertook the attempt to mathematically describe this hypothesis.

As pointed out by I. S. Lots & L. Stone (2008) there are a number of studies which are at odds with Helmholtz's hypothesis.

Firstly, as shown by Plomp and Levelt (1965) that no roughness can be experienced for intervals of pure tones larger than 3 semitones although the consonance perception varies greatly.

Secondly, Schellenberg & Trehub (1994) demonstrated that sequentially played pure tones produce a preference for simple interval ratios although without any roughness effect.

Thirdly, Peretz et al. (2001) & Tramo et al. (2001) showed that patients with auditory cortex lesions cannot perform consonance/dissonance in the fashion their uninjured counterparts can implying that consonance/dissonance perception is situated in the brain rather than in the inner ear.

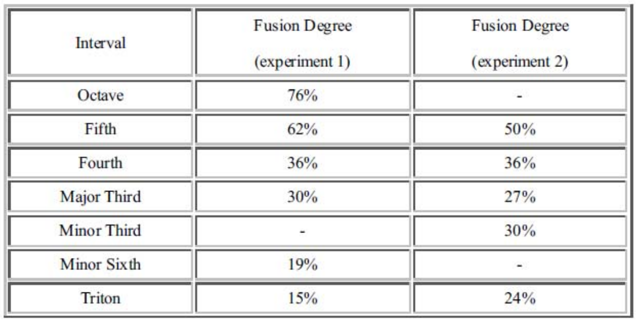
Fourthly, the EEG responses of participants that the evoked auditory responses were highest for intervals of a fifths, as shown by Itoh et al. (2003) adding further support to the hypothesis that consonance perception is a higher brain function and not a phenomenon located within the inner ear.

However, these authors do not mention the study which might be considered to be the strongest counter-evidence and this is Husmann's study from 1953 where he asked participants during an experiment to listen to intervals via bin-aural headsets with one pitch produced for the left and the other for the right ear thus preventing the occurrence of beats or roughness. However, the participants still were able to discriminate between consonant and dissonant intervals.

We conclude this section, that an overwhelming amount of evidence exists demonstrating that Helmholtz's hypothesis is unfounded and hence to be dismissed.

6. FUSION THEORY

It was Carl Stumpf (1883) who radically changed the approach to the phenomenon of consonance/dissonance by leaving behind the ratio debate as well as any acoustical explanation. In a series of experiments he asked participants to listen to a number of intervals and to state whether they were hearing one tone or two tone. Stumpf's argument was simply that the more consonant an interval of two tones was the more it would be perceived as just one tone. Stumpf called this the Verschmelzungsgrad which we translate as the fusion degree. We list the results below:



Interval	Fusion Degree (experiment 1)	Fusion Degree (experiment 2)
Octave	76%	-
Fifth	62%	50%
Fourth	36%	36%
Major Third	30%	27%
Minor Third	-	30%
Minor Sixth	19%	-
Triton	15%	24%

Table 2: The fusion degree of intervals according to Stumpf's experiments of a number of intervals

As much as there are questions about the method as employed by Stumpf such that he would exclude anyone from the experiment who had some musical knowledge and that not all intervals are included, the results are a fair indication that his approach is promissing. Particularly, the results of experiment 1, that is the order of the consonance degree of the intervals is supported by the data as produced by Hofmann-Engl (1990).

7. COGNITIVE THEORY

As much as there has been talking within the camp of Terhardt (1974) about consonance and dissonance to be related somehow to his concept of virtual pitch, this talk appears never to have been translated into a testable model. Hence the author will refrain from a further discussion.

Apparently, the only cognitive and testable model on consonance/dissonance has been forward by Hofmann-Engl (1990). It is based upon his virtual pitch model and the base hypothesis is: The easier the spectrum of virtual bass notes to a given chord is, the more consonant the chord will be. Here, the sonance factor was introduced. The virtual pitch algorithm is given as:

$$V(t) = \frac{\sum_{i=1}^{n} w_s(s_i) w_p(s_i)}{n} \qquad (4)$$

where $V(t)$ is the strength of the virtual tone t, $w_s(s_i)$ the spectral weight of the ith subharmonic of the chord, $w_p(s_i)$ the weight of the ith subharmonic according to the position of the tone within the chord, n the number of tones the chord consists of and the constant $c = 6\ Hh$

and the sonance model as:

$$S(ch) = \frac{n}{\sum_{i=1}^{n}\sqrt{\frac{1}{i}}}\,\frac{V_{\max}}{k + \frac{m^2}{k}\sqrt{1-\left(\frac{v_{p_{\max}}}{c_p}\right)^2}} \qquad (5)$$

where $S(ch)$ is the sonance (with unit Sh = Schouten) of the chord ch, v_{max} is the virtuality of the strongest root, $k = 6\ Hh/Sh$, m is the number of virtual pitches produced by the chord ch, v_{pmax} is the virtuality of the strongest root in percent (= v_{max} divided by the sum of all virtual pitches of the chord ch), $c = 0.223$ (the maximal limit the strongest root can fetch), n the number of tones the chord ch consists of and i the ith tone of the chord ch.

The model has been implemented within an applet which can be found at:

http://www.chameleongroup.org.uk/software/piano.html

Over the last two decades, this model has been tested in a number of context.

Firstly, it was tested against three groups of participants (Hofmann-Engl, 1990). These were: Secondary pupils, undergraduate music students and postgraduate music students (ca. 100 participants). The correlation between predicted data and measured data was $r^2 = 0.62$ ($p < 0.02$).

Secondly, the model then was tested in the context of contemporary composing (Hofmann-Engl, 1999). In particularly, the model was used in the context of virtual tonality.

Hofmann-Engl (2004) then used the models in order to produce a musical analysis of three 20th century composers. The analysized composition are Schoenberg's op. 19.2, Bartok's fourths and Szymanowski's Etudes op. 33.6. It appears that the application of the two models produced analytical material which otherwise would not have been obtained.

In 2006 then, Hofmann-Engl compared this virtual pitch model with the temporal model on virtual pitch within an experimental setting. The data as obtained within this experiment showed that the temporal pattern model produced erratic material while Hofmann-Engl"s model was fairly stable and clearly superior to the temporary pattern model.

Making further use of this virtual pitch/sonance model, Hofmann-Engl (2008a) was able to provide overwhelming evidence that Wagner copied the entire Tristan chord passage from Chopin.

Finally, Hofmann-Engl (2008b) demonstrated during ICMPC 10 that his virtual pitch/sonance model is in support of Riemann's chord classification system.

8. CONCLUSION & OUTLOOK

Taking a stroll through history, we found that while early attempts to tackle the issue of consonance/dissonance were based on number ratios (pure mathematics), the issue then became more of a physical/acoustical one, during the 19th century it became psychological and in the 20th century, apparently for the first time, a cognitive model was developed.

It seems more than likely that this cognitive model, as it stands now, will be replaced by far more complex models in the future. However, it seems unlikely that such models will not be based upon this model and will be similar to it.

9. REFERENCES

Bailhache, P. "Music translated into Mathematics: Leonhard Euler". In: Proceedings of Problems of Translation in the 18thCentury, Nantes, 1997

Descartes, R. "Musicae Compendium", Utrecht, 1650

Euler, L. "Tentamen novae theoriae musicae ex certissimis harmoniae principiis dilucide expositae", Petropoli: Ex typographia Academiae scientiarum,1739

Galilei, G. "Discorsi e dimostrazioni matematiche, intorno a due nuove scienze", Venizia, 1638

Hawkins, J. "General History of the Science and Practice of Music," London, 1776

Helmholtz, H. " On the sensation of tone as a physiological basis for the theory of music", Dover Publications, New York, 1877

Hofmann-Engl, L. “Music and Mathematics", online publication at: http://www.chameleongroup.org.uk/research/music_and_mathematics/music_and_mathematics_part_1.html, 2009

Hofmann-Engl, L. “Virtual Pitch and the Classification of Chords in Minor and Major Keys.” In: Proceedings of ICMPC 10, Sapporo, Japan, 2008

Hofmann-Engl, L. “The Tristan chord in context.” Chameleongroup - online publication , 2008a

Hofmann-Engl, L. “ A comparison between the temporal and pattern approach to virtual pitch applied to the root detection of chords.” In: Proceedings of ICMPC 9, Bologna, Italy, 2006

Hofmann-Engl, L. “ Virtual pitch and its application to contemporary harmony analysis.” Chameleongroup - online publication, 2004

Hofmann-Engl, L. “Virtual Pitch and pitch salience.” In: Proceedings of the VI Brazilian Symposium on Computer Music at PUC Rio de Janeiro, Brazil, 1999

Hofmann-Engl, L. “Sonanz and Virtualität”, MA thesis, TU Berlin (unpublished), 1990

Husmann, H. “*Vom Wesen der Konsonanz* .“ Heidelberg, 1953

Itoh, K., Suwazono, S. & Nakada, N. “Cortical processing of musical consonance: an evoked potential study.” In: *Neuroreport* 14**,** 2303-2306., 2003

Kepler, J. “Harmonices Mundi”, Gottfried Tamachius Tambach Bibl. Francof, 1619

Lots, I. S. & Stone, L. “Perception of musical consonance and dissonance: an outcome of neural synchronisation”, In: Interface 5, Journal of the Royal Society, 2008

Peretz, I., Blood, A. J., Penhune V. & Zatorre, R. “Cortical deafness to dissonance.” In: *Brain* **124**, 928-940, 2001

Plomp, R. & Levelt, W.J.M. “Tonal consonance and critical bandwidth,” In: *Acoust. Soc. Am.* 38, 548-560, 1965

Schellenberg, E.G. & Trehub, S.E., “Frequency ratios and the perception of tone patterns.” In: *Psychonomic Bulletin & Review 1*, 191-201, 1994

Stanley, T. “History of Philosophy”, Humphrey Moseley and Thomas Dring, London, 1655

Tramo, M.J., Cariani, P. A., Delgutte, B. & Braida, L.D. “Neurobiological foundations for the theory of harmony in western tonal music.” In: Annals of the New York Academy of Sciences 930, 92-116., 2001

Stumpf, C. “Tonsychologie.” Leipzig , 1883 & 1890

Terhardt, E. “On the perception of periodic sound fluctuations (roughness).” *Acustica*, 30(4): 201-213, 1974